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# General massive one-loop off-shell three-point functions

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**Abstract**

In this work we compute the most general massive one-loop off-shell three-point vertex in $D$-dimensions, where the masses, external momenta and exponents of propagators are arbitrary. This follows our previous paper in which we have calculated several new hypergeometric series representations for massless and massive (with equal masses) scalar one-loop three-point functions, in the negative dimensional approach.

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## 1. Introduction

In perturbative quantum field theories, obtaining analytic results for the calculation of Feynman loop integrals with massless or massive internal particles is often a hard task. Even more so if one is considering external particles off the mass-shell condition. Undoubtedly, in spite of the technical complexities attached to such calculations, great advances in this field have been accomplished [1–19], and the inherent difficulties overcome to the extent that now several two-loop and even higher order Feynman integrals are already known. In some cases one can generalize simple results to the case where one has an arbitrary number of loops [20–22].

In the two-loop arena, Glover and collaborators studied several integrals pertaining to $2 \to 2$ scattering [1, 23] and references therein. They completed the whole task. Bern, Dixon and collaborators also tackled such integrals in order to study other processes such as $2 \to 3, 2 \to 4$ and light by light scattering [8, 24, 25]. Recently, Smirnov, in the three-loop arena, presented the result for a scalar massless triple-box using Mellin–Barnes approach [26].

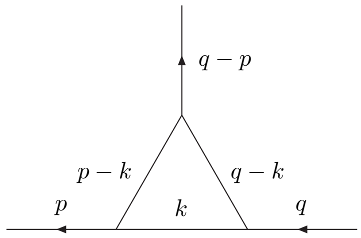


**Figure 1.** General massive off-shell triangle diagram with masses $m_1$, $m_2$ and $m_3$ for the virtual particles.

Thus, the present situation calls for the search and application of sophisticated techniques to make manageable diagrams with growing complexities leading to multi-loop integrals with a hefty load of algebraic manipulations and mathematical tools. In this scenario, we can witness considerable advances taking place in the calculation of those multi-loop integrals. On the other hand, some of the 'simpler' diagrams and associated loop integrals have not yet been fully studied; the case of the triangle diagram being one such example. This was the subject of our previous paper where we considered a one-loop triangle integral with one, two and three massive particles in the intermediate states (with equal masses), and arbitrary exponents of propagators [27]. We did show that there are dozens of generalized hypergeometric series that represent such integrals, but just a few of them are known in the literature [28].

Now we generalize those results in order to calculate scalar three-point integrals with arbitrary masses and external momenta. Our results are given in terms of generalized hypergeometric functions, most of them not known in the literature.

The outline for our paper is as follows: in section 2 we solve the most general triangle integral and present several particular cases of interest. Section 3 is devoted to our conclusions, and in the appendices we put all the tables of results, and the definitions of hypergeometric series we used throughout our paper.

## 2. One-loop three-point function

In this section, we calculate using NDIM the scalar integral of the one-loop triangle diagram (see figure 1) with two independent external momenta. Particular cases such as the massless case, and cases of one, two and three equal masses are not considered here because these are treated in our previous work [27]. Rather, here we consider other limiting cases such as when one has a vanishing exponent of a given propagator. Having said this, we have that the three-point function is given by the following scalar integral:

$$\begin{aligned} J^{(3)} &= J^{(3)}(a, b, c, D, p, q, m_1, m_2, m_3) \\ &= \int \frac{\mathrm{d}^D k}{\left[k^2 - m_1^2\right]^a \left[(k-p)^2 - m_2^2\right]^b \left[(k-q)^2 - m_3^2\right]^c}. \end{aligned} \tag{1}$$

To implement the NDIM method we start by considering the corresponding Gaussian integral

$$\begin{aligned} I &= I(p, q, m_1, m_2, m_3) \\ &= \int \mathrm{d}^D k \exp\left\{-\alpha\left(k^2 - m_1^2\right) - \beta\left[(k-p)^2 - m_2^2\right] - \gamma\left[(k-q)^2 - m_3^2\right]\right\} \end{aligned} \tag{2}$$

which can be rewritten in the form

$$I = \sum_{a,b,c=0}^{\infty} (-1)^{a+b+c} \frac{\alpha^a \beta^b \gamma^c}{a!b!c!} \int \mathrm{d}^D k \left[k^2 - m_1^2\right]^a \left[(k-p)^2 - m_2^2\right]^b \left[(k-q)^2 - m_3^2\right]^c \tag{3}$$

$$= \sum_{a,b,c=0}^{\infty} (-1)^{a+b+c} \frac{\alpha^a \beta^b \gamma^c}{a!b!c!} J^{(3)}(-a, -b, -c, D, p, q, m_1, m_2, m_3). \tag{4}$$

Note that after carrying out the analytic continuation to negative values of $a$, $b$ and $c$ in (4) we have that $J^{(3)}(-a, -b, -c, p, q, m_1, m_2, m_3)$ is just the sought after $D$-dimensional massive Feynman integral for the three-point function in (1).

Performing the $D$-dimensional Gaussian integral (2), we have

$$J = \pi^{D/2} (-1)^{-a-b-c} a!b!c! \sum_{j_1,\ldots,j_9=0}^{\infty} \frac{\Gamma(1 - j_1 - j_2 - j_3 - D/2)}{j_7! j_8! j_9!}$$

$$\times \frac{(-p^2)^{j_1}}{j_1!} \frac{(-q^2)^{j_2}}{j_2!} \frac{(-r^2)^{j_3}}{j_3!} \frac{\left(m_1^2\right)^{j_4}}{j_4!} \frac{\left(m_2^2\right)^{j_5}}{j_5!} \frac{\left(m_3^2\right)^{j_6}}{j_6!} \tag{5}$$

with the following constraint equations:

$$a = j_1 + j_2 + j_4 + j_7 \tag{6}$$

$$b = j_1 + j_3 + j_5 + j_8 \tag{7}$$

$$c = j_2 + j_3 + j_6 + j_9 \tag{8}$$

$$\frac{D}{2} = -j_1 - j_2 - j_3 - j_7 - j_8 - j_9 \tag{9}$$

where the constraints (6)–(8) come from comparing the powers of $\alpha, \beta$ and $\gamma$ with the sum in the indices $j_7$, $j_8$ and $j_9$. The last one, (9), is related to the polynomial expansion $(\alpha + \beta + \gamma)^{-D/2 - j_1 - j_2 - j_3}$.

Now, since a system of four linear algebraic equations with nine variables can only be solved if we leave five free indices, the final result will be given in terms of a five-fold series. These remaining sums can be constructed in 126 different ways, each with five free indices chosen. This number comes out quickly from the combinatorics

$$C_9^5 = \frac{9!}{5!4!} = 126. \tag{10}$$

Among these 126 solutions, 45 have trivial solutions (the determinant of the system is zero) while 12 vanish due to Pochhammer's factors in the coefficients of the series of the type

$$(0)_{a+D/2} = \frac{\Gamma(a + D/2)}{\Gamma(0)} = 0 \tag{11}$$

so that only 69 are non-trivial and independent. They are given in terms of four hypergeometric and hypergeometric-type series of the powers of the external momenta and internal masses (see appendices A and C).

In order to make our notation definitions and usage of symbols clearer, we pick a specific solution to show how the result is written down in the corresponding tables in the appendices. From tables 1 and 2 of appendix B we read the solution number 63 as follows:

$$J^{(3)} = J^3(a, b, c, p, q, r, m_1, m_2, m_3, D)$$

$$= \pi^{D/2} \left(-m_3^2\right)^{a+b+c+D/2} (D/2)_{a+b} (-c)_{-a-b-D/2}$$

$$\times \Psi_3 \left[ \begin{array}{c} -\sigma, -a, -b \\ 1 - \sigma + c, D/2 \end{array} \middle| -\frac{p^2}{m_3^2}; \frac{q^2}{m_3^2}; \frac{r^2}{m_3^2}; \frac{m_1^2}{m_3^2}; \frac{m_2^2}{m_3^2} \right] \tag{12}$$

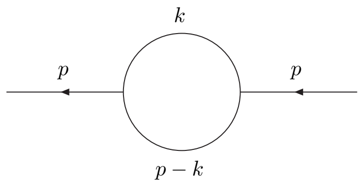


**Figure 2.** General massive off-shell two-point function.

where the hypergeometric function $\Psi_3$ is defined in appendix A and is valid in the kinematic region where $m_3 \neq 0$ and $p^2 < m_3^2, q^2 < m_3^2, r^2 = (q-p)^2 < m_3^2, m_1^2 < m_3^2, m_2^2 < m_3^2$.

Coefficients of solutions contain the Pochhammer symbol defined by

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}. \tag{13}$$

## 2.1. One-loop two-point function

Here we present the two-point function obtained from the Feynman diagram depicted (see figure 2) as a particular case of the three-point function. To achieve this we take $c = 0$ in (1), so that

$$\begin{aligned} J^{(2)} &= J^{(2)}(a, b, D, p, m_1, m_2) \\ &= \int \frac{\mathrm{d}^D k}{\left[k^2 - m_1^2\right]^a \left[(k-p)^2 - m_2^2\right]^b}. \end{aligned} \tag{14}$$

Letting the $c$ exponent be zero also implies that $j_2 = j_3 = j_6 = j_9 = 0$ in (5) and (8). In this case, we have now three constraint equations, namely (6), (7) and (9), and five variables. From this, the combinatorial analysis gives us $C_5^3 = 10$ different possibilities for the systems of linear algebraic equations. Three of them have vanishing determinant, so that we are left with seven non-trivial solutions. These are grouped into three sets, one with three solutions and two with two solutions each, according to the kinematical configuration of the variables defined by the external momenta (see appendix B). Performing the analytic continuation to $D > 0$ and $a, b < 0$ we get the three sets of solutions for the relevant Feynman integral given by

$$J^{(2)} = J_1 + J_2 + J_3 \tag{15}$$

$$J^{(2)} = J_4 + J_5 \tag{16}$$

$$J^{(2)} = J_6 + J_7 \tag{17}$$

where $J_1, J_2, \ldots, J_7$ are listed in tables 1 and 2 of appendix B. The last solution above, i.e. $J^{(2)} = J_6 + J_7$, reads

$$\begin{aligned} J^{(2)} = \pi^{D/2}\left(-m_1^2\right)^{a+D/2}\left(-m_2^2\right)^b(-a)_{-D/2} F_4\left[\begin{array}{l} D/2, -b \\ D/2, 1+a+D/2 \end{array} \middle| \frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}\right] \\ + \pi^{D/2}\left(-m_2^2\right)^{a+b+D/2}(-b)_{-a-D/2}(D/2)_a F_4\left[\begin{array}{l} -a, -a-b-D/2 \\ D/2, 1-a-D/2 \end{array} \middle| \frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}\right]. \end{aligned} \tag{18}$$

All the sets of solutions in (15)–(17) agree with the results previously known in the literature [28].

## 3. Conclusion

In this paper we have used the NDIM approach to evaluate the general massive one-loop triangle diagram. Diagrams of this type are relevant to the study of vertex corrections in QED and QCD, scalar electrodynamics, electroweak interactions and so on. We have obtained 69 solutions, each one pertaining to a specific distinct kinematical region, that show us all the different ways in which the three-point function may be expressed. These results are represented by hypergeometric-type functions of five variables. Particular cases of our results can be seen in [27, 28]. The two-point function obtained as a limiting case $c = 0$ from our results is in agreement with [28]. Therefore, the NDIM has revealed itself as a good alternative technique to the computation of Feynman integrals.

## Appendix A. Hypergeometric functions used

The Appel hypergeometric function and the hypergeometric-type functions of five variables, used in this paper, satisfy the differential equations shown in [27] and are listed below.

$$F_4\left[\begin{array}{c|c} x_1, x_2 & \\ x_3, x_4 & \end{array} z_1; z_2\right] = \sum_{j_1, j_2=0}^{\infty} \frac{(x_1)_{j_1+j_2}(x_2)_{j_1+j_2}}{(x_3)_{j_1}(x_4)_{j_2}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!}$$

$$\Psi_1\left[\begin{array}{c|c} x_1, x_2, x_3 & \\ x_4, x_5 & \end{array} z_1; z_2; z_3; z_4; z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2-j_4}(x_3)_{j_1+j_3-j_5}}{(x_4)_{j_1-j_4-j_5}(x_5)_{j_1+j_2+j_3-j_4-j_5}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!} \frac{z_4^{j_4}}{j_4!} \frac{z_5^{j_5}}{j_5!}$$

$$\Psi_2\left[\begin{array}{c|c} x_1, x_2, x_3 & \\ x_4, x_5 & \end{array} z_1; z_2; z_3; z_4; z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_5}(x_2)_{j_3+j_4+j_5}(x_3)_{j_1+j_2-j_4}}{(x_4)_{j_2-j_3-j_4}(x_5)_{j_1+j_3+j_5}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!} \frac{z_4^{j_4}}{j_4!} \frac{z_5^{j_5}}{j_5!}$$

$$\Psi_3\left[\begin{array}{c|c} x_1, x_2, x_3 & \\ x_4, x_5 & \end{array} z_1; z_2; z_3; z_4; z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3+j_4+j_5}(x_2)_{j_1+j_2+j_4}(x_3)_{j_1+j_3+j_5}}{(x_4)_{j_1+j_4+j_5}(x_5)_{j_1+j_2+j_3}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!} \frac{z_4^{j_4}}{j_4!} \frac{z_5^{j_5}}{j_5!}$$

$$\Psi_4\left[\begin{array}{c|c} x_1, x_2, x_3 & \\ x_4, x_5 & \end{array} z_1; z_2; z_3; z_4; z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_4-j_5}(x_3)_{-j_2+j_3+j_4}}{(x_4)_{j_1-j_3-j_5}(x_5)_{-j_2+j_4-j_5}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!} \frac{z_4^{j_4}}{j_4!} \frac{z_5^{j_5}}{j_5!}.$$

The set of parameters $x_i$ and variables $z_i$ for the two- and three-point functions are listed in appendices B and C, respectively.

## Appendix B. One-loop two-point solutions

The expressions for each of the solutions to the one-loop two-point function, $J_n^{(2)} = J_n^{(2)}(a, b, D, m_1, m_2)$, where $n = 1, 2, \ldots, 7$, are given by $J_n^{(2)} = D_n F_4$, where the Appel hypergeometric function $F_4$ is given in appendix A and the coefficients $D_n$ are shown in table 1, whereas the parameters and variables of the $F_4$ function are listed in table 2.

**Table 1.** Coefficients for solutions of one-loop two-point functions.

| $n$ | $D_n$ |
|---|---|
| 1 | $\pi^{D/2}(q^2)^b\left(-m_1^2\right)^{a+D/2}(-a)_{-D/2}$ |
| 2 | $\pi^{D/2}(q^2)^a\left(-m_2^2\right)^{b+D/2}(-b)_{-D/2}$ |
| 3 | $\pi^{D/2}(q^2)^{a+b+D/2}(-a)_{-b-D/2}(-b)_{2b+D/2}(a+b+D)_{-b-D/2}$ |
| 4 | $\pi^{D/2}\left(-m_1^2\right)^a\left(-m_2^2\right)^{b+D/2}(-b)_{-D/2}$ |
| 5 | $\pi^{D/2}\left(-m_1^2\right)^{a+b+D/2}(-a)_{-b-D/2}(D/2)_b$ |
| 6 | $\pi^{D/2}\left(-m_1^2\right)^{a+D/2}\left(-m_2^2\right)^b(-a)_{-D/2}$ |
| 7 | $\pi^{D/2}\left(-m_2^2\right)^{a+b+D/2}(-b)_{-a-D/2}(D/2)_a$ |

**Table 2.** Parameters and variables for hypergeometric functions of two-point functions.

| $n$ | $x_1, x_2, x_3, x_4$ | $z_1, z_2$ |
|---|---|---|
| 1 | $1-b-D/2, -b, 1+a+D/2, 1-b-D/2$ | $\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}$ |
| 2 | $1-a-D/2, -a, 1-a-D/2, 1+b+D/2$ | $\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}$ |
| 3 | $1-a-b-D, -a-b-D/2, 1-a-D/2, 1-b-D/2$ | $\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}$ |
| 4 | $-a, D/2, D/2, 1+b+D/2$ | $\frac{p^2}{m_1^2}; \frac{m_2^2}{m_1^2}$ |
| 5 | $-b, -a-b-D/2, D/2, 1-b-D/2$ | $\frac{p^2}{m_1^2}; \frac{m_2^2}{m_1^2}$ |
| 6 | $D/2, -b, D/2, 1+a+D/2$ | $\frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}$ |
| 7 | $-a, -a-b-D/2, D/2, 1-a-D/2$ | $\frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}$ |

## Appendix C. One-loop three-point solutions

The expressions for each of the solutions to the one-loop three-point function, $J_n^{(3)} = J_n^{(3)}(a, b, c, D, m_1, m_2, m_3)$, where $n = 1, 2, \ldots, 69$, are given by $J_n^{(3)} = D_n \Psi_l$, with $l = 1, 2, 3, 4$. The five-fold series $\Psi_l$ are defined in appendix A, the coefficients $D_n$ are shown in table 3 and the parameters and variables of $\Psi_l$ are given in table 4. The relation between $n$ and $l$ is given by (consider $\sigma = a + b + c + D/2$)

| $n$ | $1, \ldots, 27$ | $28, \ldots, 51$ | $52, \ldots, 63$ | $64, \ldots, 69$ |
|---|---|---|---|---|
| $l$ | 1 | 2 | 3 | 4 |

**Table 3.** Coefficients for solutions of one-loop three-point functions.

| $n$ | $D_n$ |
|---|---|
| 1 | $\pi^{D/2}(r^2)^{-a-D/2}(p^2)^{\sigma-c}(q^2)^{\sigma-b}(-a)_{2a+D/2}(-b)_{-a-D/2}(-c)_{-a-D/2}$ |
| 2 | $\pi^{D/2}(q^2)^{-b-D/2}(p^2)^{\sigma-c}(r^2)^{\sigma-a}(-b)_{2b+D/2}(-a)_{-b-D/2}(-c)_{-b-D/2}$ |
| 3 | $\pi^{D/2}(p^2)^{-c-D/2}(q^2)^{\sigma-b}(r^2)^{\sigma-a}(-c)_{2c+D/2}(-a)_{-c-D/2}(-b)_{-c-D/2}$ |
| 4 | $\pi^{D/2}(p^2)^{a-c}(q^2)^{c}\left(-m_2^2\right)^{\sigma-a}(-a)_c(-b)_{-c-D/2}$ |
| 5 | $\pi^{D/2}(q^2)^{a-b}(p^2)^{b}\left(-m_3^2\right)^{\sigma-a}(-a)_b(-c)_{-b-D/2}$ |
| 6 | $\pi^{D/2}(p^2)^{b-c}(r^2)^{c}\left(-m_1^2\right)^{\sigma-b}(-b)_c(-a)_{-c-D/2}$ |
| 7 | $\pi^{D/2}(r^2)^{b-a}(p^2)^{a}\left(-m_3^2\right)^{\sigma-b}(-b)_a(-c)_{-a-D/2}$ |
| 8 | $\pi^{D/2}(q^2)^{c-b}(r^2)^{b}\left(-m_1^2\right)^{\sigma-c}(-a)_{-b-D/2}(-c)_b$ |
| 9 | $\pi^{D/2}(r^2)^{c-a}(q^2)^{a}\left(-m_2^2\right)^{\sigma-c}(-b)_{-a-D/2}(-c)_a$ |
| 10 | $\pi^{D/2}(p^2)^{-c-D/2}\left(-m_1^2\right)^{\sigma-b}\left(-m_2^2\right)^{\sigma-a}(-c)_{2c+D/2}(-a)_{-c-D/2}(-b)_{-c-D/2}$ |
| 11 | $\pi^{D/2}(q^2)^{-b-D/2}\left(-m_1^2\right)^{\sigma-c}\left(-m_3^2\right)^{\sigma-a}(-b)_{2b+D/2}(-a)_{-b-D/2}(-c)_{-b-D/2}$ |
| 12 | $\pi^{D/2}(r^2)^{-a-D/2}\left(-m_2^2\right)^{\sigma-c}\left(-m_3^2\right)^{\sigma-b}(-a)_{2a+D/2}(-b)_{-a-D/2}(-c)_{-a-D/2}$ |
| 13 | $\pi^{D/2}(p^2)^{b}\left(-m_1^2\right)^{a-b}\left(-m_3^2\right)^{\sigma-a}(-a)_b(-c)_{-b-D/2}$ |
| 14 | $\pi^{D/2}(p^2)^{a}\left(-m_2^2\right)^{b-a}\left(-m_3^2\right)^{\sigma-b}(-b)_a(-c)_{-a-D/2}$ |
| 15 | $\pi^{D/2}(q^2)^{c}\left(-m_1^2\right)^{a-c}\left(-m_2^2\right)^{\sigma-a}(-a)_c(-b)_{-c-D/2}$ |
| 16 | $\pi^{D/2}(q^2)^{a}\left(-m_2^2\right)^{\sigma-c}\left(-m_3^2\right)^{c-a}(-b)_{-a-D/2}(-c)_a$ |
| 17 | $\pi^{D/2}(r^2)^{c}\left(-m_1^2\right)^{\sigma-b}\left(-m_2^2\right)^{b-c}(-a)_{-c-D/2}(-b)_c$ |
| 18 | $\pi^{D/2}(r^2)^{b}\left(-m_1^2\right)^{\sigma-c}\left(-m_3^2\right)^{c-b}(-a)_{-b-D/2}(-c)_b$ |
| 19 | $\pi^{D/2}(p^2)^{\sigma-c}\left(-m_3^2\right)^{c}\frac{(-b)_{2b+D/2}(-a)_{-b-D/2}}{(a+D/2)_{b+D/2}}$ |
| 20 | $\pi^{D/2}(q^2)^{\sigma-b}\left(-m_2^2\right)^{b}\frac{(-c)_{2c+D/2}(-a)_{-c-D/2}}{(a+D/2)_{c+D/2}}$ |
| 21 | $\pi^{D/2}(r^2)^{\sigma-a}\left(-m_1^2\right)^{a}\frac{(-c)_{2c+D/2}(-b)_{-c-D/2}}{(b+D/2)_{c+D/2}}$ |
| 22 | $\pi^{D/2}(p^2)^{b}\left(-m_3^2\right)^{\sigma-b}\frac{(-a)_{2a+D/2}(-c)_{-a-D/2}}{(b-a)_{a+D/2}}$ |
| 23 | $\pi^{D/2}(p^2)^{a}\left(-m_3^2\right)^{\sigma-a}\frac{(-b)_{2b+D/2}(-c)_{-b-D/2}}{(a-b)_{b+D/2}}$ |
| 24 | $\pi^{D/2}(q^2)^{c}\left(-m_2^2\right)^{\sigma-c}\frac{(-a)_{2a+D/2}(-b)_{-a-D/2}}{(-a+c)_{a+D/2}}$ |
| 25 | $\pi^{D/2}(q^2)^{a}\left(-m_2^2\right)^{\sigma-a}\frac{(-c)_{2c+D/2}(-b)_{-c-D/2}}{(a-c)_{c+D/2}}$ |
| 26 | $\pi^{D/2}(r^2)^{c}\left(-m_1^2\right)^{\sigma-c}\frac{(-b)_{2b+D/2}(-a)_{-b-D/2}}{(c-b)_{b+D/2}}$ |
| 27 | $\pi^{D/2}(r^2)^{b}\left(-m_1^2\right)^{\sigma-b}\frac{(-c)_{2c+D/2}(-a)_{-c-D/2}}{(b-c)_{c+D/2}}$ |
| 28 | $\pi^{D/2}(q^2)^{c}(p^2)^{\sigma-c}\frac{(-a)_{-b-D/2}(-b)_{2b+c+D/2}}{(a+D/2)_{b+c+D/2}}$ |
| 29 | $\pi^{D/2}(p^2)^{b}(q^2)^{\sigma-b}\frac{(-a)_{-c-D/2}(-c)_{b+2c+D/2}}{(a+D/2)_{b+c+D/2}}$ |
| 30 | $\pi^{D/2}(r^2)^{c}(p^2)^{\sigma-c}\frac{(-b)_{-a-D/2}(-a)_{2a+c+D/2}}{(b+D/2)_{a+c+D/2}}$ |
| 31 | $\pi^{D/2}(p^2)^{a}(r^2)^{\sigma-a}\frac{(-b)_{-c-D/2}(-c)_{a+2c+D/2}}{(b+D/2)_{a+c+D/2}}$ |
| 32 | $\pi^{D/2}(q^2)^{\sigma-b}(r^2)^{b}\frac{(-c)_{-a-D/2}(-a)_{2a+b+D/2}}{(c+D/2)_{a+b+D/2}}$ |
| 33 | $\pi^{D/2}(q^2)^{a}(r^2)^{\sigma-a}\frac{(-c)_{-b-D/2}(-b)_{a+2b+D/2}}{(c+D/2)_{a+b+D/2}}$ |
| 34 | $\pi^{D/2}(p^2)^{b}\left(-m_1^2\right)^{a+D/2}\left(-m_3^2\right)^{c}(-a)_{-D/2}$ |

**Table 3.** (Continued.)

| $n$ | $D_n$ |
|---|---|
| 35 | $\pi^{D/2}(p^2)^a\left(-m_2^2\right)^{b+D/2}\left(-m_3^2\right)^c(-b)_{-D/2}$ |
| 36 | $\pi^{D/2}(q^2)^c\left(-m_1^2\right)^{a+D/2}\left(-m_2^2\right)^b(-a)_{-D/2}$ |
| 37 | $\pi^{D/2}(q^2)^a\left(-m_3^2\right)^{c+D/2}\left(-m_2^2\right)^b(-c)_{-D/2}$ |
| 38 | $\pi^{D/2}(r^2)^c\left(-m_2^2\right)^{b+D/2}\left(-m_1^2\right)^a(-b)_{-D/2}$ |
| 39 | $\pi^{D/2}(r^2)^b\left(-m_3^2\right)^{c+D/2}\left(-m_1^2\right)^a(-c)_{-D/2}$ |
| 40 | $\pi^{D/2}(p^2)^b\left(-m_1^2\right)^{\sigma-b}(b+D/2)_c(-a)_{-c-D/2}$ |
| 41 | $\pi^{D/2}(p^2)^a\left(-m_2^2\right)^{\sigma-a}(a+D/2)_c(-b)_{-c-D/2}$ |
| 42 | $\pi^{D/2}(q^2)^c\left(-m_1^2\right)^{\sigma-c}(c+D/2)_b(-a)_{-b-D/2}$ |
| 43 | $\pi^{D/2}(q^2)^a\left(-m_3^2\right)^{\sigma-a}(a+D/2)_b(-c)_{-b-D/2}$ |
| 44 | $\pi^{D/2}(r^2)^c\left(-m_2^2\right)^{\sigma-c}(c+D/2)_a(-b)_{-a-D/2}$ |
| 45 | $\pi^{D/2}(r^2)^b\left(-m_3^2\right)^{\sigma-b}(b+D/2)_a(-c)_{-a-D/2}$ |
| 46 | $\pi^{D/2}\left(-m_2^2\right)^b\left(-m_1^2\right)^{\sigma-b}(D/2)_c(-a)_{-c-D/2}$ |
| 47 | $\pi^{D/2}\left(-m_1^2\right)^a\left(-m_2^2\right)^{\sigma-a}(D/2)_c(-b)_{-c-D/2}$ |
| 48 | $\pi^{D/2}\left(-m_3^2\right)^c\left(-m_1^2\right)^{\sigma-c}(D/2)_b(-a)_{-b-D/2}$ |
| 49 | $\pi^{D/2}\left(-m_1^2\right)^a\left(-m_3^2\right)^{\sigma-a}(D/2)_b(-c)_{-b-D/2}$ |
| 50 | $\pi^{D/2}\left(-m_3^2\right)^c\left(-m_2^2\right)^{\sigma-c}(D/2)_a(-b)_{-a-D/2}$ |
| 51 | $\pi^{D/2}\left(-m_2^2\right)^b\left(-m_3^2\right)^{\sigma-b}(D/2)_a(-c)_{-a-D/2}$ |
| 52 | $\pi^{D/2}(p^2)^b(q^2)^c\left(-m_1^2\right)^{a+D/2}(-a)_{-D/2}$ |
| 53 | $\pi^{D/2}(p^2)^a(r^2)^c\left(-m_2^2\right)^{b+D/2}(-b)_{-D/2}$ |
| 54 | $\pi^{D/2}(q^2)^a(r^2)^b\left(-m_3^2\right)^{c+D/2}(-c)_{-D/2}$ |
| 55 | $\pi^{D/2}(p^2)^\sigma\frac{(-b)_{2b+c+D/2}(-a)_{-b-c-D/2}}{(a+c+D/2)_{b+D/2}}$ |
| 56 | $\pi^{D/2}(q^2)^\sigma\frac{(-c)_{b+2c+D/2}(-a)_{-b-c-D/2}}{(a+b+D/2)_{c+D/2}}$ |
| 57 | $\pi^{D/2}(r^2)^\sigma\frac{(-c)_{a+2c+D/2}(-b)_{-a-c-D/2}}{(a+b+D/2)_{c+D/2}}$ |
| 58 | $\pi^{D/2}\left(-m_1^2\right)^{a+D/2}\left(-m_2^2\right)^b\left(-m_3^2\right)^c(-a)_{-D/2}$ |
| 59 | $\pi^{D/2}\left(-m_2^2\right)^{b+D/2}\left(-m_1^2\right)^a\left(-m_3^2\right)^c(-b)_{-D/2}$ |
| 60 | $\pi^{D/2}\left(-m_3^2\right)^{c+D/2}\left(-m_1^2\right)^a\left(-m_2^2\right)^b(-c)_{-D/2}$ |
| 61 | $\pi^{D/2}\left(-m_1^2\right)^\sigma(D/2)_{b+c}(-a)_{-b-c-D/2}$ |
| 62 | $\pi^{D/2}\left(-m_2^2\right)^\sigma(D/2)_{a+c}(-b)_{-a-c-D/2}$ |
| 63 | $\pi^{D/2}\left(-m_3^2\right)^\sigma(D/2)_{a+b}(-c)_{-a-b-D/2}$ |
| 64 | $\pi^{D/2}(q^2)^{\sigma-b}(p^2)^{-c-D/2}\left(-m_2^2\right)^{\sigma-a}(-a)_{-c-D/2}(-b)_{-c-D/2}(-c)_{2c+D/2}$ |
| 65 | $\pi^{D/2}(p^2)^{\sigma-c}(q^2)^{-b-D/2}\left(-m_3^2\right)^{\sigma-a}(-a)_{-b-D/2}(-c)_{-b-D/2}(-b)_{2b+D/2}$ |
| 66 | $\pi^{D/2}(r^2)^{\sigma-a}(p^2)^{-c-D/2}\left(-m_1^2\right)^{\sigma-b}(-a)_{-c-D/2}(-b)_{-c-D/2}(-c)_{2c+D/2}$ |
| 67 | $\pi^{D/2}(p^2)^{\sigma-c}(r^2)^{-a-D/2}\left(-m_3^2\right)^{\sigma-b}(-b)_{-a-D/2}(-c)_{-a-D/2}(-a)_{2a+D/2}$ |
| 68 | $\pi^{D/2}(r^2)^{\sigma-a}(q^2)^{-b-D/2}\left(-m_1^2\right)^{\sigma-c}(-a)_{-b-D/2}(-c)_{-b-D/2}(-b)_{2b+D/2}$ |
| 69 | $\pi^{D/2}(q^2)^{\sigma-b}(r^2)^{-a-D/2}\left(-m_2^2\right)^{\sigma-c}(-b)_{-a-D/2}(-c)_{-a-D/2}(-a)_{2a+D/2}$ |

**Table 4.** Parameters and variables for hypergeometric functions of three-point functions.

| $n$ | $x_1, x_2, x_3; x_4, x_5$ | $z_1, z_2, z_3, z_4, z_5$ |
|---|---|---|
| 1 | $1-\sigma-D/2, -\sigma+c, -\sigma+b; 1-a-D/2, 1-\sigma-D/2$ | $-\frac{m_1^2 r^2}{p^2 q^2}; \frac{m_2^2}{p^2}; \frac{m_3^2}{q^2}; -\frac{p^2}{r^2}; -\frac{q^2}{r^2}$ |
| 2 | $1-\sigma-D/2, -\sigma+c, -\sigma+a; 1-b-D/2, 1-\sigma-D/2$ | $-\frac{m_2^2 q^2}{p^2 r^2}; \frac{m_1^2}{p^2}; \frac{m_3^2}{r^2}; -\frac{p^2}{q^2}; -\frac{r^2}{q^2}$ |
| 3 | $1-\sigma-D/2, -\sigma+b, -\sigma+a; 1-c-D/2, 1-\sigma-D/2$ | $-\frac{m_3^2 p^2}{q^2 r^2}; \frac{m_1^2}{q^2}; \frac{m_2^2}{r^2}; -\frac{q^2}{p^2}; -\frac{r^2}{p^2}$ |
| 4 | $-c, -\sigma+a, a+D/2; a+D/2, 1+a-c$ | $-\frac{p^2 r^2}{m_2^2 q^2}; \frac{m_3^2 p^2}{m_2^2 q^2}; \frac{p^2}{q^2}; -\frac{m_1^2}{p^2}; -\frac{m_2^2}{p^2}$ |
| 5 | $-b, -\sigma+a, a+D/2; a+D/2, 1+a-b$ | $-\frac{q^2 r^2}{m_3^2 p^2}; \frac{m_2^2 q^2}{m_3^2 p^2}; \frac{q^2}{p^2}; -\frac{m_1^2}{q^2}; -\frac{m_3^2}{q^2}$ |
| 6 | $-c, -\sigma+b, b+D/2; b+D/2, 1-c+b$ | $-\frac{p^2 q^2}{m_1^2 r^2}; \frac{m_3^2 p^2}{m_1^2 r^2}; \frac{p^2}{r^2}; -\frac{m_1^2}{p^2}; -\frac{m_2^2}{p^2}$ |
| 7 | $-a, -\sigma+b, b+D/2; b+D/2, 1-a+b$ | $-\frac{q^2 r^2}{m_3^2 p^2}; \frac{m_1^2 r^2}{m_3^2 p^2}; \frac{r^2}{p^2}; -\frac{m_3^2}{r^2}; -\frac{m_2^2}{r^2}$ |
| 8 | $-b, -\sigma+c, c+D/2; c+D/2, 1-b+c$ | $-\frac{p^2 q^2}{m_1^2 r^2}; \frac{m_2^2 q^2}{m_1^2 r^2}; \frac{q^2}{r^2}; -\frac{m_1^2}{q^2}; -\frac{m_3^2}{q^2}$ |
| 9 | $-a, -\sigma+c, c+D/2; c+D/2, 1-a+c$ | $-\frac{p^2 r^2}{m_2^2 q^2}; \frac{m_1^2 r^2}{m_2^2 q^2}; \frac{r^2}{q^2}; -\frac{m_2^2}{r^2}; -\frac{m_3^2}{r^2}$ |
| 10 | $-c, -\sigma+b, -\sigma+a; 1-c-D/2, -c$ | $-\frac{m_3^2 p^2}{m_1^2 m_2^2}; \frac{q^2}{m_1^2}; \frac{r^2}{m_2^2}; -\frac{m_1^2}{p^2}; -\frac{m_2^2}{p^2}$ |
| 11 | $-b, -\sigma+c, -\sigma+a; 1-b-D/2, -b$ | $-\frac{m_2^2 q^2}{m_1^2 m_3^2}; \frac{p^2}{m_1^2}; \frac{r^2}{m_3^2}; -\frac{m_1^2}{q^2}; -\frac{m_3^2}{q^2}$ |
| 12 | $-a, -\sigma+c, -\sigma+b; 1-a-D/2, -a$ | $-\frac{m_1^2 r^2}{m_2^2 m_3^2}; \frac{p^2}{m_2^2}; \frac{q^2}{m_3^2}; -\frac{m_2^2}{r^2}; -\frac{m_3^2}{r^2}$ |
| 13 | $-b, 1-b-D/2, -\sigma+a; 1-b-D/2, 1-b+a$ | $-\frac{m_1^2 m_2^2}{m_3^2 p^2}; \frac{m_1^2}{p^2}; \frac{m_1^2 r^2}{m_3^2 p^2}; -\frac{q^2}{m_1^2}; -\frac{m_3^2}{m_1^2}$ |
| 14 | $-a, 1-a-D/2, -\sigma+b; 1-a-D/2, 1-a+b$ | $-\frac{m_1^2 m_2^2}{m_3^2 p^2}; \frac{m_2^2}{p^2}; \frac{m_2^2 q^2}{m_3^2 p^2}; -\frac{r^2}{m_2^2}; -\frac{m_3^2}{m_2^2}$ |
| 15 | $-c, 1-c-D/2, -\sigma+a; 1-c-D/2, 1-c+a$ | $-\frac{m_1^2 m_3^2}{m_2^2 q^2}; \frac{m_1^2}{q^2}; \frac{m_1^2 r^2}{m_2^2 q^2}; -\frac{p^2}{m_1^2}; -\frac{m_2^2}{m_1^2}$ |
| 16 | $-a, 1-a-D/2, -\sigma+c; 1-a-D/2, 1-a+c$ | $-\frac{m_1^2 m_3^2}{m_2^2 q^2}; \frac{m_3^2}{q^2}; \frac{m_3^2 p^2}{m_2^2 q^2}; -\frac{r^2}{m_3^2}; -\frac{m_2^2}{m_3^2}$ |
| 17 | $-c, 1-c-D/2, -\sigma+b; 1-c-D/2, 1+b-c$ | $-\frac{m_2^2 m_3^2}{m_1^2 r^2}; \frac{m_2^2}{r^2}; \frac{m_2^2 q^2}{m_1^2 r^2}; -\frac{p^2}{m_2^2}; -\frac{m_1^2}{m_2^2}$ |
| 18 | $-b, 1-b-D/2, -\sigma+c; 1-b-D/2, 1-b+c$ | $-\frac{m_2^2 m_3^2}{m_1^2 r^2}; \frac{m_3^2}{r^2}; \frac{m_3^2 p^2}{m_1^2 r^2}; -\frac{q^2}{m_3^2}; -\frac{m_1^2}{m_3^2}$ |
| 19 | $-c, b+D/2, a+D/2; 1+\sigma-c, a+b+D$ | $-\frac{p^2}{m_3^2}; \frac{q^2}{m_3^2}; \frac{r^2}{m_3^2}; -\frac{m_2^2}{p^2}; -\frac{m_1^2}{p^2}$ |
| 20 | $-b, c+D/2, a+D/2; 1+\sigma-b, a+c+D$ | $-\frac{q^2}{m_2^2}; \frac{p^2}{m_2^2}; \frac{r^2}{m_2^2}; -\frac{m_3^2}{q^2}; -\frac{m_1^2}{q^2}$ |
| 21 | $-a, c+D/2, b+D/2; 1+\sigma-a, b+c+D$ | $-\frac{r^2}{m_1^2}; \frac{p^2}{m_1^2}; \frac{q^2}{m_1^2}; -\frac{m_3^2}{r^2}; -\frac{m_2^2}{r^2}$ |
| 22 | $-b, 1-b-D/2, a+D/2; 1+\sigma-b, 1-b+a$ | $-\frac{m_3^2}{p^2}; \frac{m_2^2}{p^2}; \frac{r^2}{p^2}; -\frac{q^2}{m_3^2}; -\frac{m_1^2}{m_3^2}$ |
| 23 | $-a, 1-a-D/2, b+D/2; 1+\sigma-a, 1-a+b$ | $-\frac{m_3^2}{p^2}; \frac{m_1^2}{p^2}; \frac{q^2}{p^2}; -\frac{r^2}{m_3^2}; -\frac{m_2^2}{m_3^2}$ |
| 24 | $-c, 1-c-D/2, a+D/2; 1+\sigma-c, 1-c+a$ | $-\frac{m_2^2}{q^2}; \frac{m_3^2}{q^2}; \frac{r^2}{q^2}; -\frac{p^2}{m_2^2}; -\frac{m_1^2}{m_2^2}$ |
| 25 | $-a, 1-a-D/2, c+D/2; 1+\sigma-a, 1-a+c$ | $-\frac{m_2^2}{q^2}; \frac{m_1^2}{q^2}; \frac{p^2}{q^2}; -\frac{r^2}{m_2^2}; -\frac{m_3^2}{m_2^2}$ |
| 26 | $-c, 1-c-D/2, b+D/2; 1+\sigma-c, 1+b-c$ | $-\frac{m_1^2}{r^2}; \frac{m_3^2}{r^2}; \frac{q^2}{r^2}; -\frac{p^2}{m_1^2}; -\frac{m_2^2}{m_1^2}$ |
| 27 | $-b, 1-b-D/2, c+D/2; 1+\sigma-b, 1-b+c$ | $-\frac{m_1^2}{r^2}; \frac{m_2^2}{r^2}; \frac{p^2}{r^2}; -\frac{q^2}{m_1^2}; -\frac{m_3^2}{m_1^2}$ |

**Table 4.** (Continued.)

| $n$ | $x_1, x_2, x_3, x_4, x_5$ | $z_1, z_2, z_3, z_4, z_5$ |
|---|---|---|
| 28 | $1-\sigma-D/2, -c, -\sigma+c; 1-a-D/2, 1-\sigma+a$ | $\frac{m_2^2}{p^2}; \frac{m_1^2}{p^2}; -\frac{r^2}{q^2}; \frac{p^2}{q^2}; \frac{m_3^2}{q^2}$ |
| 29 | $1-\sigma-D/2, -b, -\sigma+b; 1-a-D/2, 1-\sigma+a$ | $\frac{m_3^2}{q^2}; \frac{m_1^2}{q^2}; -\frac{r^2}{p^2}; \frac{q^2}{p^2}; \frac{m_2^2}{p^2}$ |
| 30 | $1-\sigma-D/2, -c, -\sigma+c; 1-b-D/2, 1-\sigma+b$ | $\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}; -\frac{q^2}{r^2}; \frac{p^2}{r^2}; \frac{m_3^2}{r^2}$ |
| 31 | $1-\sigma-D/2, -a, -\sigma+a; 1-b-D/2, 1-\sigma+b$ | $\frac{m_3^2}{r^2}; \frac{m_2^2}{r^2}; -\frac{q^2}{p^2}; \frac{r^2}{p^2}; \frac{m_1^2}{p^2}$ |
| 32 | $1-\sigma-D/2, -b, -\sigma+b; 1-c-D/2, 1-\sigma+c$ | $\frac{m_1^2}{q^2}; \frac{m_3^2}{q^2}; -\frac{p^2}{r^2}; \frac{q^2}{r^2}; \frac{m_2^2}{r^2}$ |
| 33 | $1-\sigma-D/2, -a, -\sigma+a; 1-c-D/2, 1-\sigma+c$ | $\frac{m_2^2}{r^2}; \frac{m_3^2}{r^2}; -\frac{p^2}{q^2}; \frac{r^2}{q^2}; \frac{m_1^2}{q^2}$ |
| 34 | $-c, -b, b+D/2; b+D/2, 1+a+D/2$ | $\frac{m_1^2}{m_3^2}; \frac{q^2}{m_3^2}; -\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}, \frac{m_1^2 r^2}{m_3^2 p^2}$ |
| 35 | $-c, -a, a+D/2; a+D/2, 1+b+D/2$ | $\frac{m_2^2}{m_3^2}; \frac{r^2}{m_3^2}; -\frac{m_2^2}{p^2}; \frac{m_1^2}{p^2}; \frac{m_2^2 q^2}{m_3^2 p^2}$ |
| 36 | $-b, -c, c+D/2; c+D/2, 1+a+D/2$ | $\frac{m_1^2}{m_2^2}; \frac{p^2}{m_2^2}; -\frac{m_1^2}{q^2}; \frac{m_3^2}{q^2}; \frac{m_1^2 r^2}{m_2^2 q^2}$ |
| 37 | $-b, -a, a+D/2; a+D/2, 1+c+D/2$ | $\frac{m_3^2}{m_2^2}; \frac{r^2}{m_2^2}; -\frac{m_3^2}{q^2}; \frac{m_1^2}{q^2}; \frac{m_3^2 p^2}{m_2^2 q^2}$ |
| 38 | $-a, -c, c+D/2; c+D/2, 1+b+D/2$ | $\frac{m_2^2}{m_1^2}; \frac{p^2}{m_1^2}; -\frac{m_2^2}{r^2}; \frac{m_3^2}{r^2}; \frac{m_2^2 q^2}{m_1^2 r^2}$ |
| 39 | $-a, -b, b+D/2; b+D/2, 1+c+D/2$ | $\frac{m_3^2}{m_1^2}; \frac{q^2}{m_1^2}; -\frac{m_3^2}{r^2}; \frac{m_2^2}{r^2}; \frac{m_3^2 p^2}{m_1^2 r^2}$ |
| 40 | $-c, -b, -\sigma+b; b+D/2, 1-\sigma+a$ | $\frac{m_3^2}{m_1^2}; \frac{q^2}{m_1^2}; -\frac{m_2^2}{p^2}; \frac{m_1^2}{p^2}; \frac{r^2}{p^2}$ |
| 41 | $-c, -a, -\sigma+a; a+D/2, 1-\sigma+b$ | $\frac{m_3^2}{m_2^2}; \frac{r^2}{m_2^2}; -\frac{m_1^2}{p^2}; \frac{m_2^2}{p^2}; \frac{q^2}{p^2}$ |
| 42 | $-b, -c, -\sigma+c; c+D/2, 1-\sigma+a$ | $\frac{m_2^2}{m_1^2}; \frac{p^2}{m_1^2}; -\frac{m_3^2}{q^2}; \frac{m_1^2}{q^2}; \frac{r^2}{q^2}$ |
| 43 | $-b, -a, -\sigma+a; a+D/2, 1-\sigma+c$ | $\frac{m_2^2}{m_3^2}; \frac{r^2}{m_3^2}; -\frac{m_1^2}{q^2}; \frac{m_3^2}{q^2}; \frac{p^2}{q^2}$ |
| 44 | $-a, -c, -\sigma+c; c+D/2, 1-\sigma+b$ | $\frac{m_1^2}{m_2^2}; \frac{p^2}{m_2^2}; -\frac{m_3^2}{r^2}; \frac{m_2^2}{r^2}; \frac{q^2}{r^2}$ |
| 45 | $-a, -b, -\sigma+b; b+D/2, 1-\sigma+c$ | $\frac{m_1^2}{m_3^2}; \frac{q^2}{m_3^2}; -\frac{m_2^2}{r^2}; \frac{m_3^2}{r^2}; \frac{p^2}{r^2}$ |
| 46 | $-b, -c, c+D/2; 1+\sigma-b, D/2$ | $\frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}; -\frac{q^2}{m_1^2}; \frac{m_3^2}{m_1^2}; \frac{r^2}{m_2^2}$ |
| 47 | $-a, -c, c+D/2; 1+\sigma-a, D/2$ | $\frac{p_2^2}{m_1^2}; \frac{m_2^2}{m_1^2}; -\frac{r^2}{m_2^2}; \frac{m_3^2}{m_2^2}; \frac{q^2}{m_1^2}$ |
| 48 | $-c, -b, b+D/2; 1+\sigma-c, D/2$ | $\frac{q^2}{m_3^2}; \frac{m_1^2}{m_3^2}; -\frac{p^2}{m_1^2}; \frac{m_2^2}{m_1^2}, \frac{r^2}{m_3^2}$ |
| 49 | $-a, -b, b+D/2; 1+\sigma-a, D/2$ | $\frac{q^2}{m_1^2}; \frac{m_3^2}{m_1^2}; -\frac{r^2}{m_3^2}; \frac{m_2^2}{m_3^2}; \frac{p^2}{m_1^2}$ |
| 50 | $-c, -a, a+D/2; 1+\sigma-c, D/2$ | $\frac{r^2}{m_3^2}; \frac{m_2^2}{m_3^2}; -\frac{p^2}{m_2^2}; \frac{m_1^2}{m_2^2}; \frac{q^2}{m_3^2}$ |
| 51 | $-b, -a, a+D/2; 1+\sigma-b, D/2$ | $\frac{r^2}{m_2^2}; \frac{m_3^2}{m_2^2}; -\frac{q^2}{m_3^2}; \frac{m_1^2}{m_3^2}; \frac{p^2}{m_2^2}$ |
| 52 | $1-\sigma+a, -b, -c; 1+a+D/2, 1-\sigma+a$ | $-\frac{m_1^2 r^2}{p^2 q^2}; \frac{m_2^2}{p^2}; \frac{m_3^2}{q^2}; \frac{m_1^2}{p^2}; \frac{m_1^2}{q^2}$ |
| 53 | $1-\sigma+b, -a, -c; 1+b+D/2, 1-\sigma+b$ | $-\frac{m_2^2 q^2}{p^2 r^2}; \frac{m_1^2}{p^2}; \frac{m_3^2}{r^2}; \frac{m_2^2}{p^2}; \frac{m_2^2}{r^2}$ |
| 54 | $1-\sigma+c, -a, -b; 1+c+D/2, 1-\sigma+c$ | $-\frac{m_3^2 p^2}{q^2 r^2}; \frac{m_1^2}{q^2}; \frac{m_2^2}{r^2}; \frac{m_3^2}{q^2}; \frac{m_3^2}{r^2}$ |
| 55 | $-\sigma, 1-\sigma-D/2, -c; 1-\sigma+a, 1-\sigma+b$ | $-\frac{m_3^2}{p^2}; \frac{m_1^2}{p^2}; \frac{q^2}{p^2}; \frac{m_2^2}{p^2}; \frac{r^2}{p^2}$ |

**Table 4.** (Continued.)

| $n$ | $x_1, x_2, x_3, x_4, x_5$ | $z_1, z_2, z_3, z_4, z_5$ |
|---|---|---|
| 56 | $-\sigma, 1-\sigma-D/2, -b; 1-\sigma+a, 1-\sigma+c$ | $-\frac{m_2^2}{q^2}; \frac{m_1^2}{q^2}; \frac{p^2}{q^2}; \frac{m_3^2}{q^2}; \frac{r^2}{q^2}$ |
| 57 | $-\sigma, 1-\sigma-D/2, -a; 1-\sigma+b, 1-\sigma+c$ | $-\frac{m_1^2}{r^2}; \frac{m_2^2}{r^2}; \frac{p^2}{r^2}; \frac{m_3^2}{r^2}; \frac{q^2}{r^2}$ |
| 58 | $D/2, -b, -c; 1+a+D/2, D/2$ | $-\frac{m_1^2 r^2}{m_2^2 m_3^2}; \frac{p^2}{m_2^2}; \frac{q^2}{m_3^2}; \frac{m_1^2}{m_2^2}; \frac{m_1^2}{m_3^2}$ |
| 59 | $D/2, -a, -c; 1+b+D/2, D/2$ | $-\frac{m_2^2 q^2}{m_1^2 m_3^2}; \frac{p^2}{m_1^2}; \frac{r^2}{m_3^2}; \frac{m_2^2}{m_1^2}; \frac{m_2^2}{m_3^2}$ |
| 60 | $D/2, -a, -b; 1+c+D/2, D/2$ | $-\frac{m_3^2 p^2}{m_1^2 m_2^2}; \frac{q^2}{m_1^2}; \frac{r^2}{m_2^2}; \frac{m_3^2}{m_1^2}; \frac{m_3^2}{m_2^2}$ |
| 61 | $-\sigma, -b, -c; 1-\sigma+a, D/2$ | $-\frac{r^2}{m_1^2}; \frac{p^2}{m_1^2}; \frac{q^2}{m_1^2}; \frac{m_2^2}{m_1^2}; \frac{m_3^2}{m_1^2}$ |
| 62 | $-\sigma, -a, -c; 1-\sigma+b, D/2$ | $-\frac{q^2}{m_2^2}; \frac{p^2}{m_2^2}; \frac{r^2}{m_2^2}; \frac{m_1^2}{m_2^2}; \frac{m_3^2}{m_2^2}$ |
| 63 | $-\sigma, -a, -b; 1-\sigma+c, D/2$ | $-\frac{p^2}{m_3^2}; \frac{q^2}{m_3^2}; \frac{r^2}{m_3^2}; \frac{m_1^2}{m_3^2}; \frac{m_2^2}{m_3^2}$ |
| 64 | $-\sigma+a, a+D/2, c+D/2; a+D/2, 1+\sigma-b$ | $\frac{r^2}{m_2^2}; \frac{m_3^2 p^2}{m_2^2 q^2}; \frac{m_2^2}{p^2}; \frac{q^2}{p^2}; -\frac{m_1^2}{q^2}$ |
| 65 | $-\sigma+a, a+D/2, b+D/2; a+D/2, 1+\sigma-c$ | $\frac{r^2}{m_3^2}; \frac{m_2^2 q^2}{m_3^2 p^2}; \frac{m_3^2}{q^2}; \frac{p^2}{q^2}; -\frac{m_1^2}{p^2}$ |
| 66 | $-\sigma+b, b+D/2, c+D/2; b+D/2, 1+\sigma-a$ | $\frac{q^2}{m_1^2}; \frac{m_3^2 p^2}{m_1^2 r^2}; \frac{m_1^2}{p^2}; \frac{r^2}{p^2}; -\frac{m_2^2}{r^2}$ |
| 67 | $-\sigma+b, b+D/2, a+D/2; b+D/2, 1+\sigma-c$ | $\frac{q^2}{m_3^2}; \frac{m_1^2 r^2}{m_3^2 p^2}; \frac{m_3^2}{r^2}; \frac{p^2}{r^2}; -\frac{m_2^2}{p^2}$ |
| 68 | $-\sigma+c, c+D/2, b+D/2; c+D/2, 1+\sigma-a$ | $\frac{p^2}{m_1^2}; \frac{m_2^2 q^2}{m_1^2 r^2}; \frac{m_1^2}{q^2}; \frac{r^2}{q^2}; -\frac{m_3^2}{r^2}$ |
| 69 | $-\sigma+c, c+D/2, a+D/2; c+D/2, 1+\sigma-b$ | $\frac{p^2}{m_2^2}; \frac{m_1^2 r^2}{m_2^2 q^2}; \frac{m_2^2}{r^2}; \frac{q^2}{r^2}; -\frac{m_3^2}{q^2}$ |

## References


[1] Anastasiou C, Glover E W N and Oleari C 2000 *Nucl. Phys.* B **575** 416
Anastasiou C, Glover E W N and Oleari C 2000 *Nucl. Phys.* B **585** 763 (erratum)
Garland L W, Gehrmann T, Glover E W N, Koukoutsakis A and Remiddi E 2002 *Nucl. Phys.* B **642** 227

[2] Bern Z, Dixon L and Kosower D A 2000 *J. High Energy Phys.* JHEP01(2000)27

[3] Gehrmann T and Remiddi E 2001 *Nucl. Phys.* B **601** 287
Gehrmann T and Remiddi E 2000 *Nucl. Phys.* B **89** 251

[4] Chetyrkin K, Misiak M and Muenz M 1998 *Nucl. Phys.* B **518** 473

[5] Laporta S and Remiddi E 1995 *Phys. Lett.* B **356** 390
Laporta S and Remiddi E 1996 *Phys. Lett.* B **379** 283
Hughes V W and Kinoshita T 1999 *Rev. Mod. Phys.* **71** S133
Laporta S and Remiddi E 1997 *Acta Phys. Pol.* B **28** 959

[6] Yasui Y 2002 *Phys. Rev.* D **66** 094012
Yasui Y 2000 *Phys. Rev.* D **61** 094502
Del Duca V, Maltoni F and Trocsanyi Z 2002 *J. High Energy Phys.* JHEP05(2002)005
Garland L W, Gehrmann T, Glover E W N, Koukoutsakis A and Remiddi E 2002 *Nucl. Phys.* B **627** 107
Nagy Z and Trocsanyi Z 2001 *Phys. Rev. Lett.* **87** 082001
Cacciari M, Del Duca V, Frixione S and Trocsanyi Z 2001 *J. High Energy Phys.* JHEP02(2001)029

[7] Melrose D B 1965 *Nuovo Cimento* **40** 181

[8] Bern Z, Dixon L and Kosower D A 1994 *Nucl. Phys.* B **412** 751
Bern Z, Dixon L and Schmidt C 2002 *Phys. Rev.* D **66** 074018

[9] Cabral-Rosetti L G and Sanchis-Lozano M A 2002 *Proc. 2nd Int. Workshop on Graphs, Operands, Logic, Parallel Computation in Mathematical Physics (Cuautitlan, Mexico)* at press

[10] Schmidt M G and Schubert C 1996 *Phys. Rev.* D **53** 2150

[11] Suzuki A T and Schmidt A G M 2000 *Can. J. Phys.* **78** 769
[12] Tkachov F V 1981 *Phys. Lett.* B **100** 65
[13] Smirnov V A 2000 *Nucl. Phys.* B **566** 469
Smirnov V A 1999 *Phys. Lett.* B **460** 397
[14] Halliday I G and Ricotta R M 1987 *Phys. Lett.* B **193** 241
Halliday I G, Ricotta R M and Suzuki A T 1996 *Proc. XVII Brazilian Meeting of the Particles and Fields (Caxambu, Brazil)*
[15] Tausk J B 1999 *Phys. Lett.* B **469** 225
[16] Fleischer J, Smirnov V A, Frink A, Körner J, Kreimer D, Schilcher K and Tausk J B 1998 *Eur. Phys. J.* C **2** 747
[17] Weinzierl S 2000 *J. Phys.* G **26** 654
Weinzierl S and Kosower D A 1999 *Phys. Rev.* D **60** 054028
[18] Nagy Z 2002 *Phys. Rev. Lett.* **88** 122003
Bilenky M S, Rodrigo G and Santamaria A 1999 *Nucl. Phys.* **74** 53
Bilenky M S, Rodrigo G and Santamaria A 1998 *Vancouver 1998, High Energy Physics* vol 1, pp 723–7
Bernreuther W, Brandenburg A and Uwer P 2001 *Nucl. Phys.* **96** 79
Garland L W, Gehrmann T, Glover E W N, Koukoutsakis A and Remidi E 2002 *Nucl. Phys.* B **642** 227
Moch S, Uwer P and Weinzierl S 2002 *6th Zeuthen Workshop on Elementary Particle Theory (Loops and Legs in Quantum Field Theory) (Kloster Banz, Germany)*
Moch S, Uwer P and Weinzierl S 2002 *QCD 02: High-Energy Physics International Conference in Quantum Chromodynamics (Montpellier, France)*
Moch S, Uwer P and Weinzierl S 2001 *Budapest 2001, High-Energy Physics*
Maina E, Moretti S and Ross D A 2002 *Preprint* hep-ph/0210015
[19] Binoth T, Guillet J P, Heinrich G and Schubert C 2001 *Nucl. Phys.* B **615** 385
Heinrich G and Binoth T 2000 *Nucl. Phys.* **89** 246
Binoth T, Guillet J P and Heinrich G 2000 *Nucl. Phys.* B **572** 361
[20] Suzuki A T and Schmidt A G M 2000 *J. Phys.* A **33** 3713
[21] Davydychev A I and Usyukina N I 1993 *Phys. Lett.* B **305** 136
Usyukina N I 1991 *Phys. Lett.* B **267** 382
Usyukina N I 1991 *Theor. Math. Phys.* **88** 683
[22] Belokurov V V and Usyukina N I 1983 *J. Phys. A: Math. Gen.* **16** 2811
[23] Glover E W N, Tausk J B and van der Bij J J 2001 *Phys. Lett.* B **516** 33
[24] Bern Z, Dixon L and Kosower D A 1998 *Nucl. Phys.* B **513** 3
[25] Bern Z, de Freitas A, Dixon L J, Ghinculov A and Wong H L 2001 *J. High Energy Phys.* JHEP11(2001)031
[26] Smirnov V A 2002 *Phys. Lett.* B **547** 239
[27] Suzuki A T, Santos E S and Schmidt A G M 2002 *Eur. Phys. J.* C **26** 125
[28] Boos É É and Davydychev A I 1991 *Theor. Math. Phys.* **89** 1052